# ON THE GENERALIZED DERIVATIONS OF ALGEBRAS

# A B S T R A C T

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# A B S T R A C T


This thesis "*ON THE GENERALIZED DERIVATIONS OF ALGEBRAS*" has been spread over in five chapters.

**Chapter 1** gives the history of derivations starting from Isaac Newton (1642-1727) and G. W. Leibnitz (1646-1717). It tells us Geometrical and Physical significance of the derivative.

**Chapter 2** deals with Generalized Derivations in Lie and Ritt Algebra. Let $\mathcal{U}$ be an arbitrary non-associative algebra over a field $K$ and $\mathcal{GD}(\mathcal{U})$ be the set of all generalized derivations on $\mathcal{U}$.

Here we have proved that

(1) $f_1 + f_2 \in \mathcal{GD}(\mathcal{U})$

(2) Lie product $[f_1, f_2] \in \mathcal{GD}(\mathcal{U})$.
So, $\mathcal{GD}(\mathcal{U})$ becomes generalized differential algebra of $\mathcal{U}$.

(3) Jacobi identity.

(4) Generalized Leibnitz Theorem
"If Char $K = 0$ then

$$\frac{f^n(xy)}{\lfloor n} = \sum_{i=0}^{n} \left(\frac{1}{\lfloor i} f^i(x)\right) \left(\frac{1}{\lfloor n-i} D^{n-i}(y)\right)''$$

(5) "If $\mathcal{U}$ is associative or Lie Algebra then inner generalized derivations $f_a$ form an Ideal $\mathcal{J}(\mathcal{U})$ in the derivation algebra $D(\mathcal{U})$," which generalizes **Jacobson** [12, p.10].

(6) $\mathcal{GD}(\mathcal{U})$ is not closed with respect to multiplication by means of an example.
It is our attempt that $f_1 f_2$ is a generalized derivation iff $f_1$ and $f_2$ satisfy extra conditions. Also if $f_1$ depends on $f_2$, $f_2$ be an arbitrary generalized derivation then $f_1$ is also a generalized derivation.

(7) "Let $R$ be an integral domain and let $f$ be a generalized derivation on $R$ then $f$ can be extended in a unique way to a generalized derivation $F$ of the quotient field $K$ of $R$

$$\text{For} \quad \frac{x}{y} \in K \,(x, y \in R, \;\; y \neq 0)$$

$$\text{we get} \quad F\left(\frac{x}{y}\right) = \frac{f(x)y - xD(y)}{y^2}.''$$

which generalizes **Zariski** & **Samuel** [22,p.120].

(8) Let $I$ be generalized differential Ideal in a Generalized Ritt Algebra. If $a$ be any element with $a^n \in I$ then

$$(f(a))^{2n-1} \in I,$$

which generalizes **Kaplansky** [13, p.12].

(9) If $f$ is generalized derivation on $A$ then

$$f\,(\text{associator}) = 0$$

where associator is defined as in [9], $[x, y, z] = (xy)z - x(yz)$, $\forall\ x, y, z \in A$, $A$ be the any non-associative algebra.

**Chapter 3** studies the generalized Jordan derivations in prime rings of Ch $\neq$ 2. Initially we define Generalized Jordan derivation $f : A \to A$, $A$ be the prime ring of Ch $\neq$ 2 by

$$\text{(i)} \qquad f(a+b) = f(a) + f(b)$$
$$\text{(ii)} \qquad f(ab) = f(b)a + bd(a) \quad \forall \quad a, b \in A$$

where $d$ is defined as reverse derivations of $A$.

We have proved that

(1) If $A$ be a prime ring and suppose that $f$ is a non-zero generalized Jordan derivation and $d$ is a non-zero reverse derivation of $A$ then $A$ must be commutative integral domain.

(2) If $f$ is generalized Jordan Derivation of $A$ then $\forall$, $a, b \in A$

$$f(aba) = f(a)ba + ad(b)a + abd(a),$$

which generalizes **Herstein** [7, p.1106].

(3) We denote $a^b = f(ab) - f(a)b - ad(b)$ Then we have proved

$$\text{(i)} \qquad a^{b+c} = a^b + a^c$$
$$\text{(ii)} \qquad a^b = -b^a \qquad \forall \quad a, b \in A$$

(4) If $A$ is prime ring of Ch $\neq$ 2 then any generalized Jordan derivation is a generalized derivation i.e. $f(ab) = f(a)b + ad(b)$,
which is the definition of Generalized Derivation of **Havala** [6,p.1147].

Then we have redefined a Generalized Jordan Derivation on any ring. $f$ is Generalized Jordan Derivation on any ring $A$ if it satisfies the followings:

$$\begin{array}{ll} \text{(i)} & f(a+b) = f(a) + f(b) \\ \text{(ii)} & f(ab) = f(b)a + bd(a) \\ \text{(iii)} & f(aba) = f(a)b + ad(b)a + abd(a) \quad \forall \quad a, b \in A \end{array}$$

where $d$ is the reverse derivation.

(5) Let $A$ be any prime ring of Ch $= 2$ and if $A$ is not commutative Integral domain, then any generalized Jordan derivation is a generalized derivation.

(6) "If $R$ admits a $(\sigma, \tau)$ generalized derivation $f$ such that $f^2(I) = 0$, $I$ be a non-zero Ideal of 2-torsion free ring $R$ and $f$ commutes with both $\sigma, \tau$ then $f = 0$, $d = 0$," which generalizes **Mohd. Asraf and Nadeem-Ur-Rahaman** [19, p.260].

**Chapter 4** deals with the Generalized inner derivations in a ring. Let $A$ be a ring, then an additive mapping $f : A \to A$ is said to be generalized inner derivation if

$$\begin{aligned} f(xy) &= f(x)y + h_a(y) \quad \text{where} \\ h_a &: A \to A \\ & \quad y \to h_a(y) = [a, y] \end{aligned}$$

is the inner derivations $\forall\ x, y \in A$, fixed element $a \in A$. Let $\mathcal{G}_{\mathcal{I}}\mathcal{D}(A)$ be the set of all generalized inner derivation of $A$ into itself.

We have proved that

(1) If $f \in \mathcal{G}_{\mathcal{I}}\mathcal{D}(A)$ then

$$f(xyz) = f(x)yz + xh_a(yz) \quad \forall\ x, y, z \in A$$

(2) If $f$ is generalized inner derivation in semi-prime ring $A$ then $h_a$ must necessarily be a derivation, which generalizes **Havala** [6,p.1147].

(3) Let $A$ be a semi-prime ring then $\forall\ x, y \in A$

$$f(xyx) = f(x)yx + xd(y)x + xyd(x),$$

which generalizes **Herstein** [7, p.1106].

(4) If $rf(x) = 0 \; \forall \; x \in A$, $r$ be any element of $A$, $A$ being Prime ring then either $r = 0$ or $h_a = 0$.

This result generalizes **Posner** [5, p.1093].

(5) If $f \in \mathcal{G_I D(A)}$ then

$$f(xyz) = f(xy)z + xf(yz) - xf(y)z \forall \; x, y, z \in A, f \in \mathcal{G_I D}(A),$$

which generalizes **Bresar** [16, p.90].

(6) If $A$ has unity then generalized inner derivations become inner derivation and vice versa.

(7) If $f(aba) = f(a)ba \; \forall \; b \in A$ where $f$ is generalized inner derivation on prime ring $A$.

(8) if $f \in \mathcal{G_I D}(A)$ then

$$x(f(x)a + af(x)) = f(x)(ax + xa),$$

for fixed element $a \in A$.

(9) Let $K$ be non-zero Ideal of $A$, $A$ with unity satisfying

$$xy + f(xy) = yx + f(yx)$$

then

$$(1 + f(1))[x, y] + [a, [x, y]] = 0 \qquad \forall \quad x, y \in A.$$

(10) Using **Havala** [6, p.1147] def. of Generalized derivation we have proved.

(i) $d(b)a = ad(b)$

(ii) $f(ab + ba) = f(a)b + f(b)a + d(ab)$

(iii) If $A$ is without zero divisors and if $ab = 0$ then $f(ab) = f(a)b + f(b)a$

(iv) $ad(b)ba = abd(b)a \quad \forall \; a, \; b \in A$, $A$ be any prime ring of Ch $\neq 2$.

**Chapter 5** is devoted to study the generalized graded derivation. We define a linear mapping $f : A \to A$ where $A$ be any graded algerba, is generalized graded derivation if

$$f(ab) = f(a)b + (-1)^{|a|\,|f|} aD(b) \quad \forall a, b \in A$$

where $D =$ derivation on $A$.

This result generalizes **Havala** [6, p.1147] def. of Generalized Derivation and **Leibnitz** Rule. Let $\mathcal{G_R D(A)}$ be the set of all generalized graded derivation of $A$.

We have proved

(1) (a) $f_1 + f_2 \notin \mathcal{G_R D(A)}$

(b) $[f_1, f_2] \notin \mathcal{G_R D(A)}$

(2) Let $f$ be a generalized graded derivation of $A$, $A$ being the graded algebra defined by

$$f(xy) = f(x)y + (-1)^{|x|\,|f|}xD(y) \quad \forall x, y \in A.$$

If $af(x) = 0$, $a \in A$ then either $a = 0$ or $D = 0$, which generalizes **Posner** [5, p.1093].

(3) If $f$ is generalized graded derivation of $A$ then

$$f(aba) = \begin{cases} f(a)ba + aD(b)a + abD(a) & \text{if } |f| = \text{ even} \\ f(a)ba + (-1)^{|a|}aD(b)a + (-1)^{|a|\,|b|}abD(a) & \text{if } |f| = \text{ odd,} \end{cases}$$

which generalizes **Herstein** [7, p.1106].

(4) $f(a^n) = f(a^2)a^{n-2} + a^2D(a^{n-2}) \quad \forall n \geq 3$

Putting the values of $n$, we get **Havala** [6, p.1147] results.

(5) If $f \in \mathcal{G_R D}(A)$ then

$$\left((-1)^{|a|\,|b|\,|f|} - (-1)^{|a|\,|f|}\right) abD(b)a = 0.$$

(6) If $f \in \mathcal{G_R D}(A)$, $A$ be the graded algebra then

$$f(xyz) = \begin{cases} f(xy)z + xf(yz) - xf(y)z \text{ if } |f| = \text{ even} \\ f(xy)z + xf(yz) - xf(y)z + \\ \left((-1)^{|x|\,|y|} - (-1)^{|y|}\right) xyD(z) \text{ if } |f| = \text{ odd,} \end{cases}$$

which generalizes **Bresar** [16,p.90].

In the end we have given a list of research papers and books which we have used in this thesis.

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