# ABSTRACT OF PH.D. THESIS

**Title of Ph.D. Thesis:** *The Study of Some Generalized Gaussian Hypergeometric Functions.*

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The present thesis comprises of SEVEN CHAPTERS. A brief summary of the problems is presented at the beginning of each chapter and then each chapter is divided into a number of sections. Equations in every section are numbered separately. For example the small bracket (a.b.c) specified the results, in which the last figure denotes the equation number, the middle one the section and the first indicate the chapter to which it belongs. Sections, Articles, Definitions and Equations have been numbered chapter wise.

The aim of the **first chapter** (DETAILED SURVEY OF LITERATURE) is to introduce several classes of special functions, which occur rather more frequently in the study of summations, transformations and generating functions needed for the presentation of subsequent chapters. In this chapter, we have discussed different forms of Gamma function; Legendre duplication and triplication formulae; Psi function; Polygamma function $\psi^{(m)}(x)$; Complete and Incomplete Beta and Gamma functions, Prym's function; Fractional derivative; Fractional integration; Hankel's contour integral formula; A useful limit formula for certain infinite products; Pochhammer symbol; Ordinary hypergeometric function of one variable ${}_AF_B$, its convergence conditions, properties associated with well-poised series, very well-poised series, Saalschützian series, nearly poised series of first and second kinds; Wright's generalized hypergeometric function ${}_p\Psi_q,\ {}_p\Psi^\star_q$; MacRobert and Meijer functions; Ordinary bilateral and truncated hypergeometric functions; Hypergeometric summation theorems; Linear, quadratic and cubic transformations and Reduction formulae; Multiplication formulae; The $\beta$-function $\beta^{(n)}(x)$; Polylogarithm functions; Struve function; Analytic continuation formulae; Catalan's constant $G$; Hypergeometric forms and integral representation of some special functions applicable in mathematical physics; Ordinary double hypergeometric functions of Exton and Kampé de Fériet; General triple hypergeometric function of Srivastava; General quadruple hypergeometric functions of Srivastava, Pathan and Saigo; Multiple ordinary hypergeometric function of Srivastava-Daoust; Bernoulli and Euler numbers; Riemann's Zeta function; General class of polynomials, classical orthogonal polynomials and multivariable polynomials; Some series identities, expansion of inverse function and Lagrange's inversion formula; Different equivalent definitions, notations and identities for basic hypergeometric series; Continued fractions for hypergeometric function; Some interesting continued fractions of Ramanujan; Second, third and fourth order Ramanujan numbers and their integrals; Results on number theory; Lambert series; Some results on partition theory etc.

It provides a systematic introduction to most of the important special functions that commonly arise in practice and explores many of their salient properties. This chapter

is also intended to make the thesis as much self contained as possible.

In **second chapter** (SOME DEFINITE INTEGRALS AND TAU FUNCTION OF SRINIVASA RAMANUJAN), we have given simple proofs of Mellin transform of ${}_AF_B$ and six definite integrals including Ramanujan's integrals, without the application of Master theorem. The Laplace transform of Beta function given by R. P. Agarwal, is also modified here.

Ramanujan calculated the values of $\tau(1), \tau(2), \cdots\cdots, \tau(30)$, by means of the theory of elliptic functions and certain arithmetical functions such as $F_{r,s}(x)$, $\Phi_{r,s}(x)$, $E_{r,s}(n)$, $\sigma_s(n)$, Riemann's Zeta function $\zeta(n)$, Greatest integer function $[x]$, Theory of symbols $o, O$, Continued fraction, Asymptotic expansion, Some trigonometrical identities, Inequalities, Gamma function, Theory of order of error terms, Number theory, Convergence and divergence of infinite series.

We have obtained the values of $\tau(1), \tau(2), \tau(3), \cdots\cdots, \tau(36), \tau(37)$ using some algebraic techniques applied in

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x\left\{\prod_{n=1}^{\infty}(1-x^n)\right\}^{24}$$

where $\tau(n)$ is Tau function of Ramanujan.

**Chapter third** (TRUNCATED HYPERGEOMETRIC SERIES OF L.J.SLATER) is associated with the modified and correct forms of Slater's erroneous results of partial summation theorem given in a book *'Generalized Hypergeometric Functions'* published by Cambridge University Press. Using Thomae's first fundamental transformation, Hardy's transformation and Rainville limit formula for certain infinite products, we obtain some new summation theorems for non-terminating 2-balanced ${}_3F_2(1)$ series.

In the Maclaurin's expansions of $\tan z$, $\cot z$, $\operatorname{cosec} z$, $\tanh(z)$, $\coth(z)$, $\operatorname{cosech}(z)$ and $\sec z$, $\operatorname{sech}(z)$, the coefficients of $z^n$ are associated with Bernoulli numbers and Euler numbers respectively. From their Maclaurin's expansions, we are unable to obtain their corresponding hypergeometric forms.

In **chapter four** (MITTAG-LEFFLER'S PARTIAL FRACTION EXPANSIONS OF MEROMORPHIC FUNCTIONS), we shall obtain hypergeometric forms of some meromorphic functions $\frac{1}{e^z-1}$, $\sec^2 z$, $\operatorname{cosec}^2 z$, $\tan z$, $\cot z$, $\operatorname{cosec} z$, $\sec z$, $\operatorname{sech}^2(z)$, $\operatorname{cosech}^2(z)$, $\tanh(z)$, $\coth(z)$, $\operatorname{cosech}(z)$, $\operatorname{sech}(z)$, $\frac{\pi}{8z^3}\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$, $\frac{\pi}{4z}\frac{\sinh(2\pi z)-\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$ and $\frac{\pi}{4z^2}\frac{\sinh(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$, from corresponding partial fraction expansions, using a beautiful lemma $a+pn = \frac{a\left(\frac{a+p}{p}\right)_n}{\left(\frac{a}{p}\right)_n}$.

In **chapter five** (SUMMATION THEOREMS MOTIVATED BY THE WORKS OF RAMANUJAN, CHUDNOVSKY AND BORWEIN), we shall establish some hypergeometric summation theorems by giving particular values to the parameters $a$, $b$ and the argument $x$; three summation theorems for ${}_2F_3(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; x)$, three summation theorems for ${}_4F_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{a+b}{b}; 1, 1, \frac{a}{b}; x)$, two summation theorems for ${}_4F_3(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{a+b}{b}; 1, 1, \frac{a}{b}; x)$, four summation theorems for ${}_4F_3(\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{a+b}{b}; 1, 1, \frac{a}{b}; x)$ and ten summation theorems for ${}_4F_3(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{a+b}{b}; 1, 1, \frac{a}{b}; x)$. By the application of said beautiful lemma, the above summation theorems are obtained from seventeen series and three definite integrals of Ramanujan, Borwein series and Chudnovsky series.

The main object of **chapter six** (TRANSFORMATION AND REDUCTION FORMULAE INVOLVING QUARTER, HALF AND UNIT ARGUMENTS) is to obtain two general theorems on multiple series identity involving bounded sequences. The theorems, in turn, are expressed in terms of Srivastava-Daoust hypergeometric function of three variables. Certain results involving unit, half and quarter arguments associated with hypergeometric polynomials ${}_7F_6$, ${}_6F_5$, ${}_5F_4$, ${}_4F_3$, ${}_3F_2$ and ${}_4\Psi_4^{\star}$, are obtained. Some known transformations and reduction formulae of Joshi-Vyas, Baweja and Chaudhary involving generalized Gauss function, double hypergeometric functions of Kampé de Feriét and Exton are also obtained. Further many more known or new results can be obtained by specializing the parameters or the variables or both.

The **last chapter** (A HYPERGEOMETRIC APPROACH TO EXACT SOLUTIONS OF POWER LAW POTENTIALS) is associated with a problem of Physics.

In this chapter, we consider the more general force

$$f(x) = -K\, x^{2n-1}$$

where $n$ is any positive integer and $K$ is a given positive constant.

The force in above expression is proportional to odd powers of $x$ hence this force is always attractive; the motion is therefore expected to be periodic on physical grounds. The equation of motion can be written in terms of the usual integral expression. However this integral expression can not be reduced to an explicit form using elementary functions; for $n > 1$. The equation of motion is thus not written in an explicit form and to perform calculations, one has to resort to *approximate methods*. Similarly, the time period can be written as an integral; that integral expression is again not reducible to an explicit form, when $n > 1$.

**By using hypergeometric function, one can overcome this difficulty.** For any general positive integral value of $n$ in the expression for the force acting on the particle; one can write explicit expressions relating the position $x$ of the particle with time $t$. Similarly, for the time period $T$ of the periodic motion one may obtain explicit expressions.

A detailed bibliography appears at the end with the authors name in alphabetical order. References to the bibliography are numbered. The thesis include with appendices which contains reprints of a few published papers and some mathematical useful tables etc..